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*Original Citation:*

Pellerey F. (2007). *Comparison Results for Branching Processes in Random Environments*. In: [JOURNAL OF APPLIED PROBABILITY](#), pp. 142-150. - ISSN 0021-9002

*Availability:*

This version is available at : <http://porto.polito.it/1579245/> since: June 2007

*Publisher:*

Applied Probability Trust

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# Comparison results for branching processes in random environments

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## Author's version.

Published in: *Journal of Applied Probability* 44 (2008), pp. 142–150.

URL:<http://projecteuclid.org/euclid.jap/1175267168>.

## Abstract

In this note we consider branching processes whose behaviors depend on a dynamic random environment, in the sense that we assume the offsprings distributions of individuals parametrized, during time, by the realizations of a process describing the environmental evolution. We study how the variability in time of the environment modifies the variability of total population: considered two branching processes of such kind, but subjected to different environments, we provide conditions on the random environments in order to stochastically compare their marginal distributions in increasing convex sense. Weaker conditions are also provided for comparisons at every fixed time of the expected values of the two populations.

**AMS Subject Classification:** 60E15, 60G99

**Key words and phrases:** Branching processes, increasing convex order, supermodular order, concordance order.

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\*Partially supported by Italian PRIN-Cofin 2004 Stochastic Models in Mathematical Finance

# 1 Preliminaries and utility notions

Branching processes are commonly used in applied probability to model the development of populations whose members produce offsprings according to stochastic laws (see Harris (1989)). Initially introduced as a tool for specific biological problems, today the range of applications of branching processes includes molecular and cellular biology, human evolution, medicine, and other fields like physics, computer science or actuarial science (see Rolski et al. (1999), Teich and Saleh (2000) or Kimmel and Axelrod (2002), among others).

In literature the classical definition of standard branching process is the following: it is a process  $\mathbf{Z} = \{Z_n, n \in \mathbb{N}\}$  such that  $Z_0$  has a fixed known distribution and

$$Z_n = \sum_{j=1}^{Z_{n-1}} X_{j,n}, \quad n \geq 1.$$

The integer-valued random variables  $X_{j,n}$ , with  $j, n \in \mathbb{N}$ , are usually assumed to be all independent, and identically distributed for every fixed  $n$ . Typically, the value  $Z_n$  denotes the size of a population at the  $n$ -th generation (or season), while the random variable  $X_{j,n}$  represents the number of offsprings of the  $j$ -th individual at the  $n$ -th generation, with  $j, n \in \mathbb{N}$ . The independence and identically distributed assumption for the  $X_{j,n}$  means that individuals reproduce independently of each other according to some given offspring distribution.

In the literature there exist several results about stochastic comparisons for population sizes of branching processes in the case that the numbers of offsprings are independent. In order to state two of them, we recall the definition of two well-known stochastic orders (see Shaked and Shanthikumar (1994) for properties and applications of these orders).

**Definition 1.1.** *Given two non-negative random variables  $X$  and  $Y$ ,  $X$  is said to be smaller than  $Y$  in the usual stochastic order [increasing convex order] (denoted  $X \leq_{st} Y$  [ $X \leq_{icx} Y$ ]) if  $\mathbf{E}[u(X)] \leq \mathbf{E}[u(Y)]$  for all increasing [increasing convex] functions  $u$  for which previous expectations exist.*

Consider now two standard branching processes  $\mathbf{Z}_1 = \{Z_{1,n}, n \in \mathbb{N}\}$  and  $\mathbf{Z}_2 = \{Z_{2,n}, n \in \mathbb{N}\}$  defined letting  $Z_{1,0} = Z_{2,0} = 1$  a.s. and then recursively by

$$Z_{i,n} = \sum_{j=1}^{Z_{i,n-1}} X_{j,n}^i, \quad n \geq 1, i = 1, 2.$$

One can prove that  $Z_{1,n} \leq_{st} [\leq_{icx}] Z_{2,n}$  for all  $n \in \mathbb{N}$  whenever  $X_{j,n}^1 \leq_{st} [\leq_{icx}] X_{j,n}^2$  for all  $n \in \mathbb{N}$ . The first of this statements is easy to prove, while a proof for the increasing and convex comparison case may be found in Section 8 of Ross (1983).

In this paper, we are interested in generalizations of these results in the case that the offspring distribution of individuals depends on environmental conditions (see, e.g., Smith and Wilkinson (1969), Athreya and Karlin (1970) or Jagers and Zhunwei (2002) for examples of applications of branching processes defined on random environments). In particular, in this paper we focus

on studying how the variability in time of the environment modifies the variability of the total population.

To this aim, it is possible to generalize the setup above to situations in which the distribution of the numbers of offsprings depends on some random geographical or economic environment  $\Theta$ . This can be modeled as follows. Let  $\mathcal{X} \subseteq \mathbb{R}$ , and let  $\theta = \{\theta_n \in \mathcal{X}, n \in \mathbb{N} \cup \{0\}\}$  be any sequence of values in  $\mathcal{X}$ . For each  $\theta$ , let  $\mathbf{X}(\theta)$  be an infinite array of non-negative integer valued random variables parametrized by  $\theta$  as follows:

$$\mathbf{X}(\theta) = \begin{vmatrix} X_{1,0}(\theta_0) & X_{1,1}(\theta_1) & \cdots & X_{1,n}(\theta_n) & \cdots \\ X_{2,0}(\theta_0) & X_{2,1}(\theta_1) & \cdots & X_{2,n}(\theta_n) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}. \quad (1.1)$$

It will be assumed below that, for each fixed  $\theta$ , the columns of  $\mathbf{X}(\theta)$  are independent, and that, within each column, the variables are independent. Thus, if we consider only the first  $n+1$  components of  $\theta$  (i.e, if we consider the restriction  $\theta_n = (\theta_0, \theta_1, \dots, \theta_n) \in \mathcal{X}^{n+1} \subseteq \mathbb{R}^{n+1}$  of  $\theta$ ) then the restriction  $\mathbf{X}_n(\theta_n)$  of  $\mathbf{X}(\theta)$  to the first  $n+1$  columns is of the form

$$\mathbf{X}_n(\theta_n) = [\mathbf{X}_0(\theta_0), \mathbf{X}_1(\theta_1), \dots, \mathbf{X}_n(\theta_n)] = \begin{vmatrix} X_{1,0}(\theta_0) & X_{1,1}(\theta_1) & \cdots & X_{1,n}(\theta_n) \\ X_{2,0}(\theta_0) & X_{2,1}(\theta_1) & \cdots & X_{2,n}(\theta_n) \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}, \quad (1.2)$$

where, given  $\theta_n$ , the distribution of the  $k$ th column of  $\mathbf{X}_n(\theta_n)$  depends only on  $\theta_k$ ,  $k = 0, 1, \dots, n$ , and the variables in the column are independent.

Let now  $\theta = \{\theta_0, \theta_1, \dots\}$  be a sequence of values in  $\mathcal{X}$  describing the evolutions of the environment, and define, recursively, the stochastic process  $\mathbf{Z}(\theta) = \{Z_n(\theta_0, \dots, \theta_n), n \in \mathbb{N}\}$  by

$$Z_0(\theta_0) = X_{1,0}(\theta_0)$$

and

$$Z_n(\theta_0, \dots, \theta_n) = \sum_{j=1}^{Z_{n-1}(\theta_0, \dots, \theta_{n-1})} X_{j,n}(\theta_n), \quad n \geq 1. \quad (1.3)$$

In order to consider random evolutions of the environment, we can consider a sequence  $\Theta = (\Theta_0, \Theta_1, \dots)$  of random variables taking on values in  $\mathcal{X}$ . Thus, we will be interested in stochastic processes  $\mathbf{Z}(\Theta) = \{Z_n(\Theta_0, \dots, \Theta_n), n \in \mathbb{N}\}$  defined by

$$Z_0(\Theta_0) = X_{1,0}(\Theta_0)$$

and

$$Z_n(\Theta_0, \dots, \Theta_n) = \sum_{j=1}^{Z_{n-1}(\Theta_0, \dots, \Theta_{n-1})} X_{j,n}(\Theta_n), \quad n \geq 1, \quad (1.4)$$

where, for every  $j, k \in \mathbb{N}$ ,  $X_{j,k}(\Theta_k)$  is a random variable such that  $[X_{j,k}(\Theta_k) | \Theta_k = \theta] =_{st} X_{j,k}(\theta)$  (here  $=_{st}$  means equality in law).

In case of random environments having fixed identical marginal distributions it has been shown to be useful the use of some dependence orders that have been introduced in literature to compare the strength of positive dependence within two multivariate distributions (see, for example, Joe (1997), Shaked and Shanthikumar (1997) or Bäuerle and Rieder (1997)). In this paper we consider two of them, whose definitions are given here. For it, recall that a real-valued function  $\phi$  defined on  $\mathbb{R}^m$  is said to be supermodular if  $\phi(\mathbf{x} \vee \mathbf{y}) + \phi(\mathbf{x} \wedge \mathbf{y}) \geq \phi(\mathbf{x}) + \phi(\mathbf{y})$  for all  $\mathbf{x}$  and  $\mathbf{y} \in \mathbb{R}^m$  (here  $\vee$  and  $\wedge$  denote, respectively, the componentwise maximum and minimum).

**Definition 1.2.** Let  $\mathbf{X} = (X_1, X_2, \dots, X_n)$  and  $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$  be two random vectors with equal marginal distributions. Then  $\mathbf{X}$  is said to be smaller than  $\mathbf{Y}$ :

- i) in the supermodular order (denoted by  $\mathbf{X} \leq_{sm} \mathbf{Y}$ ) if  $\mathbf{E}[\phi(\mathbf{X})] \leq \mathbf{E}[\phi(\mathbf{Y})]$  for every supermodular function  $\phi$  for which previous expectations exist.
- ii) in the concordance order (denoted by  $\mathbf{X} \leq_c \mathbf{Y}$ ) if  $\mathbf{E}[\prod_{i=1}^n \phi_i(X_i)] \leq \mathbf{E}[\prod_{i=1}^n \phi_i(Y_i)]$  for every collection  $\{\phi_1, \phi_2, \dots, \phi_n\}$  of non-negative and increasing functions for which previous expectations exist.

We note that the supermodular order implies the concordance order (which is also called positive quadrant dependence order), while (unless for the case  $n = 2$ ) the reversed implications does not hold (see Müller and Scarsini (2000)), and both comparisons are interpreted in the sense as  $\mathbf{Y}$  being more positively dependent than  $\mathbf{X}$ .

We recall also the definition of the usual stochastic order in the multivariate setting, and an equivalent condition that will be used in next section.

**Definition 1.3.** Let  $\mathbf{X} = (X_1, \dots, X_n)$  and  $\mathbf{Y} = (Y_1, \dots, Y_n)$  be two  $n$ -dimensional random vectors, then  $\mathbf{X}$  is said to be smaller than  $\mathbf{Y}$  in the multivariate stochastic order (denoted by  $\mathbf{X} \leq_{st} \mathbf{Y}$ ) if  $\mathbf{E}[\phi(\mathbf{X})] \leq \mathbf{E}[\phi(\mathbf{Y})]$  for all increasing real-valued functions  $\phi$  defined on  $\mathbb{R}^n$  for which the expectations exists.

**Property 1.1.** The random vectors  $\mathbf{X}$  and  $\mathbf{Y}$  satisfy  $\mathbf{X} \leq_{st} \mathbf{Y}$  if, and only if, there exist two random vectors  $\hat{\mathbf{X}}$  and  $\hat{\mathbf{Y}}$ , defined on the same probability space, such that  $\mathbf{X} =_{st} \hat{\mathbf{X}}$ ,  $\mathbf{Y} =_{st} \hat{\mathbf{Y}}$  and  $\hat{\mathbf{X}} \leq \hat{\mathbf{Y}}$  a.s.

Finally, the following monotonicity property will be used in the next section. In the definition, the inequality  $\mathbf{u} \leq \mathbf{v}$  for two vectors  $\mathbf{u} = (u_1, u_2, \dots, u_m)$  and  $\mathbf{v} = (v_1, v_2, \dots, v_m)$  means  $u_i \leq v_i$  for all  $i = 1, 2, \dots, m$ .

**Definition 1.4.** Let  $\{\mathbf{Y}(\mathbf{p}), \mathbf{p} \in \mathcal{P} \subseteq \mathbb{R}^m, m \in \mathbb{N}\}$  be a finite or infinite family of random vectors parametrized by an  $m$ -dimensional vector of parameters  $\mathbf{p}$ . Then  $\{\mathbf{Y}(\mathbf{p}), \mathbf{p} \in \mathcal{P}\}$  is said to be stochastically increasing in  $\mathbf{p}$  if  $\mathbf{Y}(\mathbf{p}) \leq_{st} \mathbf{Y}(\mathbf{p}')$  for all  $\mathbf{p} \leq \mathbf{p}'$ .

As we have mentioned above, the purpose of this paper is to study how the variability in time of the environment influences the variability of the populations. To this aim, we consider two branching processes defined as in (1.4), but subjected to different random environments  $\boldsymbol{\Theta}_1 = (\Theta_{1,0}, \Theta_{1,1}, \dots)$  and  $\boldsymbol{\Theta}_2 = (\Theta_{2,0}, \Theta_{2,1}, \dots)$ . Motivated by the comparison results mentioned at the

beginning, we derive conditions on the environments in order to ensure stochastic comparisons of the corresponding populations. In particular, we state conditions under which the supermodular order between environments implies the increasing convex order of the populations. Also, we identify conditions under which the concordance order between environments provides comparisons of the expected values of the corresponding populations at every fixed time.

Throughout the next sections  $[X | E]$  denotes a random element whose distribution is identical to that of  $X$  conditional on the event  $E$ , and the terms "increasing" and "decreasing" are used in non-strict sense. Also, for notational convenience we define  $\sum_{j=1}^0 x_j = 0$  for every sequence of real numbers  $\{x_j, j \in \mathbb{N}\}$ .

## 2 Comparisons results

Throughout this and the next section we will make the following assumptions on the array  $\mathbf{X}(\boldsymbol{\theta})$ :

A1)  $\mathbf{X}(\boldsymbol{\theta})$  is an infinite array of non-negative integer valued random variables with independent columns of independent variables as described in (1.1);

A2) for all  $k = 0, 1, \dots$  the  $k$ -th column of  $\mathbf{X}(\boldsymbol{\theta})$  is stochastically increasing in  $\theta_k$ ;

A3) the variables in each column of  $\mathbf{X}(\boldsymbol{\theta})$  are stochastically increasing, in the sense that  $X_{j,k}(\theta_k) \leq_{st} X_{j+1,k}(\theta_k)$  for all  $j, k \in \mathbb{N}$  and  $\theta_k \in \mathcal{X}$ .

Note that, as a particular case, condition A3 above is satisfied when all the variables in each column  $\mathbf{X}_k(\theta_k)$  of  $\mathbf{X}(\boldsymbol{\theta})$  are independent and identically distributed for every fixed value of the parameter  $\theta_k$ .

It is easy to verify that, under assumptions A1–A3, the  $n$ -th population size  $Z_n(\theta_1, \dots, \theta_n)$  is stochastically increasing in  $(\theta_1, \dots, \theta_n)$ . From this fact easily follows that the total population increases in usual stochastic order as the environment stochastically increases. Actually, using Property 1.1 it is also easy to prove that, always under assumptions A1–A3, the whole process  $\mathbf{Z}(\boldsymbol{\Theta}_1)$  is stochastically smaller than the whole process  $\mathbf{Z}(\boldsymbol{\Theta}_2)$  (i.e.,  $\mathbf{E}[u(\mathbf{Z}(\boldsymbol{\Theta}_1))] \leq \mathbf{E}[u(\mathbf{Z}(\boldsymbol{\Theta}_2))]$  for all increasing functionals  $u$  such that both expectations exist) whenever the sequence  $\boldsymbol{\Theta}_1$  is stochastically smaller than the sequence  $\boldsymbol{\Theta}_2$ .

However, it is rather natural to imagine that the size of the population at any generation also depends on monotonicity and regularity properties of the environmental process. The subsequent Theorem 2.1 is motivated by this observation, and it describes how dependence properties of the process  $\boldsymbol{\Theta}$  modify, in increasing convex order sense, the distribution of  $Z_n(\Theta_1, \dots, \Theta_n)$ .

**Theorem 2.1.** *Let  $\mathbf{X}(\boldsymbol{\theta})$  be an infinite array of non-negative integer valued random variables satisfying the assumptions A1–A3, and let  $\boldsymbol{\Theta}_1 = (\Theta_{1,0}, \Theta_{1,1}, \dots)$  and  $\boldsymbol{\Theta}_2 = (\Theta_{2,0}, \Theta_{2,1}, \dots)$  be two sequences of random variables taking on values in  $\mathcal{X}$ . Assume that both  $\boldsymbol{\Theta}_1, \boldsymbol{\Theta}_2$  are independent on  $\mathbf{X}(\boldsymbol{\theta})$ . Then for every  $n \in \mathbb{N}$  the stochastic inequality*

$$(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n}) \leq_{sm} (\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n}) \quad (2.1)$$

*implies*

$$Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n}) \leq_{icx} Z_n(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n}) \quad (2.2)$$

*Proof.* First of all we will prove by induction that, for every fixed  $n \in \mathbb{N}$ , the function  $\phi(\theta_0, \dots, \theta_n) = \mathbf{E}[u(Z_n(\theta_0, \dots, \theta_n))]$  is supermodular in  $(\theta_0, \dots, \theta_n)$  whenever the function  $u$  is increasing and convex.

Since  $\phi(\theta_0, \theta_1)$  is supermodular by Theorem 2.1 in Belzunce et al. (2006), it is enough to prove that supermodularity of  $\phi(\theta_0, \dots, \theta_n)$  in  $(\theta_0, \dots, \theta_n)$  follows from supermodularity of  $\tilde{\phi}(\theta_0, \dots, \theta_{n-1}) = \mathbf{E}[\tilde{u}(Z_{n-1}(\theta_0, \dots, \theta_{n-1}))]$  in  $(\theta_0, \dots, \theta_{n-1})$  whenever the function  $\tilde{u}$  is increasing and convex.

To this aim, it suffices to show that  $\phi(\theta_0, \dots, \theta_n)$  is supermodular in any couple  $(\theta_i, \theta_j)$ ,  $0 \leq i < j \leq n$  (see, e.g., Kulik (2003)).

Let us consider first the case  $(\theta_i, \theta_n)$ ,  $0 \leq i < n$ . For it, let  $(\theta_i, \theta_n)$  and  $(\theta'_i, \theta'_n)$  be any two vectors defined on  $\mathcal{X}^2$  such that  $\theta_i \leq \theta'_i$  and  $\theta_n \leq \theta'_n$ . Observe that, since  $Z_{n-1}(\theta_1, \dots, \theta_{n-1})$  is stochastically increasing in  $(\theta_1, \dots, \theta_{n-1})$ , we can build on the same probability space the random variables  $\hat{Z}_{n-1}$  and  $\hat{Z}'_{n-1}$  such that  $\hat{Z}_{n-1} =_{st} Z_{n-1}(\theta_0, \dots, \theta_i, \dots, \theta_{n-1})$ ,  $\hat{Z}'_{n-1} =_{st} Z_{n-1}(\theta_0, \dots, \theta'_i, \dots, \theta_{n-1})$ , and

$$\hat{Z}_{n-1} \leq \hat{Z}'_{n-1} \quad \text{a.s.} \quad (2.3)$$

Thus,

$$\begin{aligned} & \phi(\theta_0, \dots, \theta'_i, \dots, \theta'_n) - \phi(\theta_0, \dots, \theta_i, \dots, \theta_n) \\ &= \mathbf{E} \left[ \mathbf{E} \left[ u \left( \sum_{j=1}^{\hat{Z}'_{n-1}} X_{j,n}(\theta'_n) \right) - u \left( \sum_{j=1}^{\hat{Z}_{n-1}} X_{j,n}(\theta'_n) \right) \middle| \hat{Z}'_{n-1}, \hat{Z}_{n-1} \right] \right] \\ &= \mathbf{E} \left[ \mathbf{E} \left[ g_{\hat{Z}'_{n-1}}^{\hat{Z}'_{n-1}}(\mathbf{X}_n(\theta'_n)) \middle| \hat{Z}'_{n-1}, \hat{Z}_{n-1} \right] \right] \\ &\geq \mathbf{E} \left[ \mathbf{E} \left[ g_{\hat{Z}'_{n-1}}^{\hat{Z}'_{n-1}}(\mathbf{X}_n(\theta_n)) \middle| \hat{Z}'_{n-1}, \hat{Z}_{n-1} \right] \right] \\ &= \mathbf{E} \left[ \mathbf{E} \left[ u \left( \sum_{j=1}^{\hat{Z}'_{n-1}} X_{j,n}(\theta_n) \right) - u \left( \sum_{j=1}^{\hat{Z}_{n-1}} X_{j,n}(\theta_n) \right) \middle| \hat{Z}'_{n-1}, \hat{Z}_{n-1} \right] \right] \\ &= \phi(\theta_0, \dots, \theta'_i, \dots, \theta_n) - \phi(\theta_0, \dots, \theta_i, \dots, \theta_n) \end{aligned}$$

where the inequality follows from (2.3), assumption A2 and the fact that the function  $g_l^m(\bar{y}) = u(\sum_{i=1}^m y_i) - u(\sum_{i=1}^l y_i)$  is an increasing function in  $\bar{y} = \{y_1, y_2, \dots\}$  whenever  $m \geq l$  and  $\bar{y}$  is any sequence of non-negative integer numbers. Thus, obviously,

$$\phi(\theta_0, \dots, \theta'_i, \dots, \theta'_n) + \phi(\theta_0, \dots, \theta_i, \dots, \theta_n) \geq \phi(\theta_0, \dots, \theta_i, \dots, \theta'_n) + \phi(\theta_0, \dots, \theta'_i, \dots, \theta_n),$$

i.e.,  $\phi(\theta_0, \dots, \theta_n)$  is supermodular in  $(\theta_i, \theta_n)$ ,  $0 \leq i < n$ .

Now we consider the case  $(\theta_i, \theta_j)$  with  $0 \leq i < j < n$ . For it, observe that the function  $\tilde{u}(z) = u(\sum_{j=1}^z y_j)$  is increasing and convex in  $z \in \mathbb{N}$  whenever  $u$  is an increasing and convex function and  $\bar{y} = \{y_1, y_2, \dots\}$  is any increasing sequence of non-negative integer numbers. Also, recall that, by inductive assumption, the function  $\tilde{\phi}(\theta_0, \dots, \theta_{n-1}) = \mathbf{E}[\tilde{u}(Z_{n-1}(\theta_0, \dots, \theta_{n-1}))]$  is supermodular in  $(\theta_0, \dots, \theta_{n-1})$  for every increasing and convex function  $\tilde{u}$ . Moreover, by assumption

A3 we can build on the same probability space the random sequence  $\hat{\mathbf{X}}_n = \{\hat{X}_{j,n}, j \in \mathbb{N}\}$  such that  $\hat{X}_{j,n}(\theta_n) =_{st} X_{j,n}(\theta_n)$  and

$$\hat{X}_{j,n}(\theta_n) \leq \hat{X}_{j+1,n}(\theta_n) \quad \text{a.s.} \quad (2.4)$$

for all  $j, n \in \mathbb{N}$ .

Thus, denoted  $\hat{\mathbf{X}}_n(\theta_n) = \{\hat{X}_{j,n}(\theta_n), j \in \mathbb{N}\}$ ,

$$\begin{aligned} & \phi(\theta_0, \dots, \theta'_i, \dots, \theta'_j, \dots, \theta_n) - \phi(\theta_0, \dots, \theta_i, \dots, \theta'_j, \dots, \theta_n) \\ &= \mathbf{E} \left[ \mathbf{E} \left[ u \left( \sum_{j=1}^{Z_{n-1}(\dots, \theta'_i, \dots, \theta'_j, \dots, \theta_n)} \hat{X}_{j,n}(\theta_n) \right) - u \left( \sum_{i=1}^{Z_{n-1}(\dots, \theta_i, \dots, \theta'_j, \dots, \theta_n)} \hat{X}_{j,n}(\theta_n) \right) \middle| \hat{\mathbf{X}}_n(\theta_n) \right] \right] \\ &= \mathbf{E} \left[ \mathbf{E} \left[ \tilde{u} (Z_{n-1}(\dots, \theta'_i, \dots, \theta'_j, \dots, \theta_n)) - \tilde{u} (Z_{n-1}(\dots, \theta_i, \dots, \theta'_j, \dots, \theta_n)) \middle| \hat{\mathbf{X}}_n(\theta_n) \right] \right] \\ &\geq \mathbf{E} \left[ \mathbf{E} \left[ \tilde{u} (Z_{n-1}(\dots, \theta'_i, \dots, \theta_j, \dots, \theta_n)) - \tilde{u} (Z_{n-1}(\dots, \theta_i, \dots, \theta_j, \dots, \theta_n)) \middle| \hat{\mathbf{X}}_n(\theta_n) \right] \right] \\ &= \mathbf{E} \left[ \mathbf{E} \left[ u \left( \sum_{j=1}^{Z_{n-1}(\dots, \theta'_i, \dots, \theta_j, \dots, \theta_n)} \hat{X}_{j,n}(\theta_n) \right) - u \left( \sum_{i=1}^{Z_{n-1}(\dots, \theta_i, \dots, \theta_j, \dots, \theta_n)} \hat{X}_{j,n}(\theta_n) \right) \middle| \hat{\mathbf{X}}_n(\theta_n) \right] \right] \\ &= \phi(\theta_0, \dots, \theta'_i, \dots, \theta_j, \dots, \theta_n) - \phi(\theta_0, \dots, \theta_i, \dots, \theta_j, \dots, \theta_n), \end{aligned}$$

where the inequality follows from remarks above on the function  $\tilde{u}$ , inequality (2.4) and subsequent supermodularity of  $\mathbf{E}[\tilde{u}(Z_{n-1}(\theta_0, \dots, \theta_{n-1})) | \hat{\mathbf{X}}_n(\theta_n)]$ .

Thus  $\phi(\theta_0, \dots, \theta_n)$  is supermodular also in  $(\theta_i, \theta_j), 0 \leq i < j < n$ , and supermodularity of  $\phi(\theta_0, \dots, \theta_n)$  in  $(\theta_0, \dots, \theta_n)$  follows.

Now we get the increasing convex comparison among population sizes  $Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})$  and  $Z_n(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n})$  just observing that, for every fixed increasing and convex function  $u$ , it holds

$$\begin{aligned} \mathbf{E}[u(Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n}))] &= \mathbf{E}[\phi(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})] \\ &\leq \mathbf{E}[\phi(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n})] = \mathbf{E}[u(Z_n(\Theta_{1,0}, \Theta_{2,1}, \dots, \Theta_{2,n}))], \end{aligned}$$

where the function  $\phi$  is defined as above. □

Under weaker assumptions one can also obtain the weaker comparisons among the expected margins of the two branching processes, as stated in the following result.

**Theorem 2.2.** *Let  $\mathbf{X}(\theta)$  satisfy assumption A1, and let  $\Theta_1 = (\Theta_{1,0}, \Theta_{1,1}, \dots)$  and  $\Theta_2 = (\Theta_{2,0}, \Theta_{2,1}, \dots)$  be two sequences of random variables taking on values in  $\mathcal{X}$ . Assume that both  $\Theta_1, \Theta_2$  are independent on  $\mathbf{X}(\theta)$ , and that  $\mathbf{E}[X_{1,k}(\theta_k)]$  is increasing in  $\theta_k$  for all  $k = 0, 1, \dots$ . Then for every  $n \in \mathbb{N}$  the stochastic inequality*

$$(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n}) \leq_c (\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n}) \quad (2.5)$$

implies

$$\mathbf{E}[Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})] \leq \mathbf{E}[Z_n(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n})].$$



*Proof.* Let  $(\theta_0, \dots, \theta_n) \in \mathcal{X}^{n+1}$  and note that  $\mathbf{E}[Z(\theta_0, \dots, \theta_n)] = \prod_{k=0}^n \mathbf{E}[X_{1,k}(\theta_k)]$  (the equality trivially comes from the fact that all the random variables  $X_{j,k}(\theta_k)$  are independent, and identically distributed for fixed values of  $k$ ). Observing that, by assumptions, every  $\mathbf{E}[X_{1,k}(\theta_k)]$  is increasing in the  $\theta_k$ , by (2.5) we get

$$\begin{aligned} \mathbf{E}[Z(\Theta_{1,0}, \dots, \Theta_{1,n})] &= \mathbf{E}[\mathbf{E}[Z_n(\Theta_{1,0}, \dots, \Theta_{1,n}) \mid (\Theta_{1,0}, \dots, \Theta_{1,n})]] \\ &= \mathbf{E}\left[\prod_{k=0}^n \mathbf{E}[X_{1,k}(\Theta_{1,k})]\right] \leq \mathbf{E}\left[\prod_{k=0}^n \mathbf{E}[X_{1,k}(\Theta_{2,k})]\right] \\ &= \mathbf{E}[\mathbf{E}[Z_n(\Theta_{2,0}, \dots, \Theta_{2,n}) \mid (\Theta_{2,0}, \dots, \Theta_{2,n})]] = \mathbf{E}[Z(\Theta_{2,0}, \dots, \Theta_{2,n})] \end{aligned}$$

i.e., the assertion.  $\square$

### 3 An example of application

Assume that the random evolutions of the environment are described by a stationary discrete-time homogeneous Markov process  $\Theta = \{\Theta_n : n \in \mathbb{N}\}$  that is stochastically monotone (i.e., such that  $[\Theta_2 \mid \Theta_1 = \theta]$  is stochastically increasing in  $\theta$ ). Using the criteria described in the previous section one can define stochastic bounds for the total population at any generation. In fact, let  $\Theta_1 = \{\Theta_{1,n} : n \in \mathbb{N}\}$  be a sequence of variables such that  $\Theta_{1,n} = \Theta_{1,0}$  a.s. for all  $n \in \mathbb{N}$  where  $\Theta_{1,0}$  has the same distribution of  $\Theta_0$  (i.e., the stationary marginal distribution of  $\Theta$ ). Then it is well-known that  $(\Theta_0, \Theta_1, \dots, \Theta_n) \leq_{sm} (\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})$  for every  $n \in \mathbb{N}$  (see, e.g., Tchen, 1980 on this inequality).

Let now  $\Theta_2 = \{\Theta_{2,n} : n \in \mathbb{N}\}$  be a sequence of independent and identically distributed variables such that  $\Theta_{2,n} =_{st} \Theta_0$  (i.e., having as distribution the stationary marginal distribution of  $\Theta$ ). It has been shown (see, e.g., Hu and Pan (2000)) that in this case it holds  $(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n}) \leq_{sm} (\Theta_0, \Theta_1, \dots, \Theta_n)$  for every  $n \in \mathbb{N}$ .

Therefore, for the branching process  $\mathbf{Z}(\Theta)$  defined as in (1.4), and subjected to an underlying stationary discrete-time homogeneous Markov process  $\Theta$ , the following two assertions hold.

**Corollary 3.1.** *Let  $\mathbf{X}(\theta)$  be an infinite array of non-negative integer valued random variables satisfying the assumptions A1–A3. If  $\mathbf{X}(\theta)$  is independent on  $\Theta$  then*

$$Z_n(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n}) \leq_{icx} Z_n(\Theta_0, \Theta_1, \dots, \Theta_n) \leq_{icx} Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})$$

for every  $n \in \mathbb{N}$ .

**Corollary 3.2.** *Let  $\mathbf{X}(\theta)$  be an infinite array of non-negative integer valued satisfying assumption A1. If  $\mathbf{E}[X_{1,k}(\theta_k)]$  is increasing in  $\theta_k$  for all  $k = 0, 1, \dots$ , and if  $\mathbf{X}(\theta)$  is independent on  $\Theta$ , then*

$$\mathbf{E}[Z_n(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n})] \leq \mathbf{E}[Z_n(\Theta_0, \Theta_1, \dots, \Theta_n)] \leq \mathbf{E}[Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})]$$

for every  $n \in \mathbb{N}$ .

The interest on these results is due to the fact that the distributions of  $Z_n(\Theta_{2,0}, \Theta_{2,1}, \dots, \Theta_{2,n})$  and of  $Z_n(\Theta_{1,0}, \Theta_{1,1}, \dots, \Theta_{1,n})$  can be calculated in closed form observing that they are nothing else than a standard branching process and a mixture of standard branching processes. Note also that if  $\Theta$  describes the behavior of the environment and the columns of  $\mathbf{X}(\theta)$  are stochastically increasing in the parameters  $\theta_k$  then the assumption that  $\Theta$  is stochastically monotone is realistic and common in applicative contexts.

Always assuming that the underlying process  $\Theta$  is a stationary discrete-time homogeneous Markov process, other interesting examples of application of the results presented in Section 2 may be provided considering Theorem 3.2 and Theorem 4.1 in Hu and Pan (2000).

## 4 Acknowledgements

I want to thank Professor Moshe Shaked and an anonymous referee for their valuable comments and suggestions that greatly improved the presentation of this paper.

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